Assignment 9

Coverage: 16.5, 16.6 in Text.

Exercises: 16.5 no 4, 8, 10, 13, 17, 19, 24, 33, 42, 48, 56; 16.6 no 4, 7, 10, 15.

Hand in 16.5 no 33, 48, 56; 16.6 no 10 by April 8.

Supplementary Problems

1. The zeros of a function $F(x, y, z) = 0$ may define a surface in space. Let $S = \{(x, y, z) : F(x, y, z) = 0\}$ where F is C^1. Suppose that $F_z \neq 0$. By Implicit Function Theorem the set S can be locally described as the graph of a function $z = \varphi(x, y)$. Suppose now $S = \{(x, y, \varphi(x, y)),\ (x, y) \in D\}$ where D is a region in the xy-plane. Derive the following surface area for S:
$$|S| = \iint_D \frac{|\nabla F|}{|F_z|}\, dA(x, y) \ .$$

2. Let $(x(t), y(t)),\ t \in [a, b]$, be a curve C parametrized by t in the first and the second quadrants. Rotate it around the x-axis to get a surface of revolution S.

   (a) Show that a parametrization of S is given by $(\alpha, t) \mapsto (x(t), y(t)\cos\alpha, y(t)\sin\alpha)$ $\alpha \in [0, 2\pi]$, and it is regular when C is regular.

   (b) Show that the surface area of S is given by
$$2\pi \int_C y(t)\, ds \ .$$

   (c) When $y = \varphi(x), x \in [a, b]$, where φ is C^1, the surface area becomes
$$2\pi \int_a^b \varphi(x)\sqrt{1 + \varphi'^2(x)}\, dx \ .$$

Exercises 16.5

More Parametrizations of Surfaces

33. a. Parametrization of an ellipsoid The parametrization $x = a \cos \theta$, $y = b \sin \theta$, $0 \leq \theta \leq 2\pi$ gives the ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Using the angles θ and ϕ in spherical coordinates, show that

$$\mathbf{r}(\theta, \phi) = (a \cos \theta \cos \phi)\mathbf{i} + (b \sin \theta \cos \phi)\mathbf{j} + (c \sin \phi)\mathbf{k}$$

is a parametrization of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.

b. Write an integral for the surface area of the ellipsoid, but do not evaluate the integral.

Surface Area for Implicit and Explicit Forms

48. Find the area of the surface $2x^{3/2} + 2y^{3/2} - 3z = 0$ above the square R: $0 \leq x \leq 1$, $0 \leq y \leq 1$, in the xy-plane.

56. Let S be the surface obtained by rotating the smooth curve $y = f(x)$, $a \leq x \leq b$, about the x-axis, where $f(x) \geq 0$.

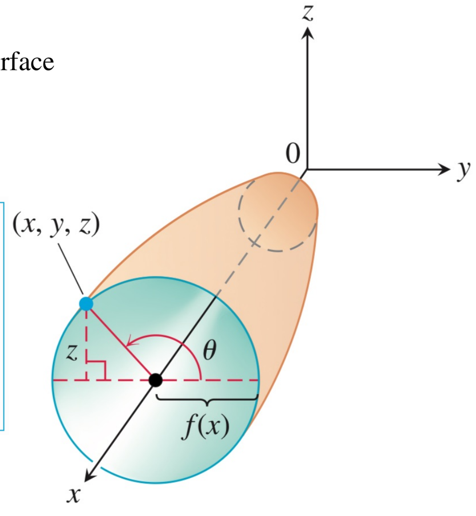
a. Show that the vector function

$$\mathbf{r}(x, \theta) = x\mathbf{i} + f(x) \cos \theta\mathbf{j} + f(x) \sin \theta\mathbf{k}$$

is a parametrization of S, where θ is the angle of rotation around the x-axis (see the accompanying figure).

b. Use Equation (4) to show that the surface area of this surface of revolution is given by

$$A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$



DEFINITION The **area** of the smooth surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \leq u \leq b, \quad c \leq v \leq d$$

is

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \tag{4}$$

Exercises 16.6

Surface Integrals of Scalar Functions

10. Integrate $G(x, y, z) = y + z$ over the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.